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A Fixed Point Theorem

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A FIXED POINT THEOREM

Walter T. Kyner

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A FIXED POINT THEOREM

Section 1

The purpose of this paper is to prove a fixed point theorem that is useful in the Diliberto perturbation theory. The (non-degenerate) perturbation theorems of G. Hufford [1] and M. Marcus [2] are implied by this result.

I wish to thank Professor S. P. Diliberto for suggesting this problem and for his continued interest in it. In addition, I wish to thank Professor L. Nirenberg for pointing out a mistake in an earlier "proof" of this theorem.

We are interested in a family of transformations defined on a subset of the space of real multiply periodic continuously differentiable m-dimensional vector valued functions of s real variables. If $f = (f_1,\ldots,f_m)$ is such a function, then $f(\theta_1,\ldots,\theta_s)$ must be periodic with period ω_k in θ_k.

We use

$$\|f\| = \max\left[\|f\|_o, \left\|\frac{\partial f}{\partial \theta_k}\right\|_o\right] ,$$

where

$$\|f\|_o = \max_i \max_\theta |f_i(\theta)| .$$

For a fixed real number γ, the transformation $g = T_\gamma(f)$ is given by

$$g(\bar{\theta}) = N(\theta)f(\theta) + W(f(\theta),\theta,\gamma) \tag{1}$$

where θ is a function of $\bar{\theta}$, f, and γ defined implicitly by

$$\bar{\theta} = V_{f\gamma}(\theta) = \theta + \pi + U(f(\theta),\theta,\gamma) . \tag{2}$$

π is a constant s-vector. $N(\theta)$ is an $m \times m$ matrix whose entries C^2 functions of θ. $U(y,\theta,\gamma)$ and $W(y,\theta,\gamma)$ are vector valued functions of $m+s+1$ real variables whose second partial derivatives with respect to y and θ are continuous in y, θ, and γ. U, W, and N are periodic in θ, We also require that

$$W(0,\theta,0) \equiv 0 ,$$

$$U(0,\theta,0) \equiv 0 ,$$

and

$$\frac{\partial W(0,\theta,0)}{\partial y} \equiv 0 .$$

The transformations used by Marcus and Hufford are of this form.

To insure that equation (2) can be solved for θ, we restrict γ and $\|f\|$ so that

$$\left(\frac{\partial \bar{\theta}_i}{\partial \theta_j}\right)$$

is non-singular.

For each γ sufficiently small, we hope to find a function that is invariant under T_γ. It is necessary to make an additional restriction on the transformations considered. Hufford (for $s = 1$) assumed that all but one of the characteristic exponents of a variational equation had non-zero real parts. This hypothesis determines the form of $N(\theta)$. For his many variable result, Marcus assumed that the norm of $N(\theta)$ was less than one. Marcus' theorem does not imply Hufford's. It will be shown below that their hypotheses imply the following:

(H): For each f and γ such that $\|f\| \leq r_o$, $|\gamma| \leq \gamma_o$, define the linear transformation $L_{f\gamma}$ by

$$[L_{f\gamma}g](\theta) = g(V_{f\gamma}(\theta)) - N(\theta)g(\theta) . \tag{3}$$

We assume that $L_{f\gamma}$ is invertible and that its inverse is uniformly bounded, i.e., $\|L_{f\gamma}^{-1}\| \leq A$ for all $\|f\| \leq r_o$. $|\gamma| \leq r_o$.

Section 2

THEOREM. If T_γ is defined as above, and if the hypotheses (H) is satisfied, then there exists γ_o such that if $|\gamma| \leq \gamma_o$, T_γ has a unique fixed point f_γ. f_γ is continuous in γ, and as $\gamma \longrightarrow 0$, $f_\gamma \longrightarrow 0$. Furthermore

$$\left\{ \frac{\partial f_\gamma}{\partial \theta_k} \right\}$$

satisfy a uniform Lipschitz condition.

PROOF. If f is invariant under T_γ, then

$$f(V_{f\gamma}) = N(\theta)f(\theta) + W(f(\theta), \theta, \gamma) . \tag{4}$$

Therefore

$$f = L_{f\gamma}^{-1} W(f, \gamma) \tag{5}$$

where

$$[W(f, \gamma)](\theta) = W(f(\theta), \theta, \gamma) .$$

It is sufficient to show that $L^{-1}W$ has a fixed point. This will be done by restricting its domain to a compact convex set. If this set and γ_0 are properly selected, then $L^{-1}W$ is continuous on this set and maps it into itself. By Schauder's theorem, $L^{-1}W$ has a fixed point. Uniqueness and continuity in γ follow from a separate computation.

Let

$$C(r, u) = \left\{ g \in C^2 \;\middle|\; \|g\| \leq r, \quad \left\| \frac{\partial g}{\partial \theta_k} \right\| \leq u, \quad k=1,\ldots,s \right\} .$$

The proof depends on the careful selection of the numbers r, u, and γ. To emphasize this a separate numbering system will be used.

(o) We pick r_0, γ_0 so that if $|r| \leq r_0$, $|\gamma| \leq \gamma_0$, then T_γ is well defined and $L_{f\gamma}^{-1}$ is uniformly bounded by A.

If f is in C(r, u) and $g = L_{f\gamma}^{-1} W(f, \gamma)$, then $\|g\| \leq A\|W(f, \gamma)\|$. We want $\|g\| \leq r$.

If we expand $W(f(\theta), \theta, \gamma)$, we get

$$\|W(f, \gamma)\| \leq \|W(0, \gamma)\| + \left\| \frac{\partial W}{\partial y} \right\| \cdot \|f\| . \tag{6}$$

To control the size of $\|\frac{\partial W}{\partial y}\|$, it is sufficient to limit $\|\frac{W_i}{y_j}\|$.

$$(7) \qquad \left\| \frac{\partial}{\partial \theta_k} \left(\frac{\partial W_i}{\partial y_j} \right) \right\|_o \leq \sum_{n=1}^{m} \left\| \frac{\partial^2 W_i}{\partial y_n \partial y_j} \right\|_o r + \left\| \frac{\partial^2 W_i}{\partial \theta_k \partial y_j} \right\|_o .$$

(i) Since $\frac{\partial^2 W_i}{\partial y_n \partial y_j}$ is uniformly bounded, and since $\frac{\partial W(0,\theta,0)}{\partial y} \equiv 0$, we can pick $r_1 \leq r_o$ and $\gamma_1 \leq \gamma_o$, so that $A\|\frac{\partial W}{\partial y}\| \leq \frac{1}{2}$.

(ii) We then pick $\gamma_2 \leq \gamma_1$, so that $A\|W(0, \gamma)\| \leq (\frac{1}{2})r_1$, if $|\gamma| \leq \gamma_2$. As we further restrict the size of r, it will be necessary to select γ so that (ii) is still satisfied.

We now have that if f is in $C(r_1,u)$ and $|\gamma| \leq \gamma_2$, then $\|L_{f\gamma}^{-1} W(f, \gamma)\| \leq r_1$. If $g = L_{f\gamma}^{-1} W(f, \gamma)$, then $L_{f\gamma} g = W(f, \gamma)$. Therefore

$$\sum_{i=1}^{s} \frac{\partial g}{\partial \theta_i} \frac{\partial V_i}{\partial \theta_k} - \frac{\partial N(\theta)}{\partial \theta_k} g(\theta) - N(\theta) \frac{\partial g(\theta)}{\partial \theta_k}$$

$$= \sum_{n=1}^{m} \frac{\partial W}{\partial y_n} \frac{\partial f_n}{\partial \theta_k} + \frac{\partial W}{\partial \theta_k} .$$

To obtain a bound on $\frac{\partial g}{\partial \theta_k}$, we form a product space and consider elements $\bar{h} = (h^1,\ldots,h^s)$. For fixed f and γ, a linear transformation $\bar{M}_{f\gamma}$ is defined on this space by

$$(9) \qquad \left[\bar{M}_{f\gamma} \bar{h} \right]^k (\theta) = \sum_{i=1}^{s} h^i(V_{f\gamma}) \frac{\partial V_i}{\partial \theta_k} - N(\theta) h^k(\theta) .$$

Defining $\bar{L}_{f\gamma}$ by $[\bar{L}_{f\gamma}] = (\delta_{ik} L_{f\gamma})$, we see that

$$(10) \quad \left[(\bar{M}_{f\gamma} - \bar{L}_{f\gamma}) \bar{h} \right]^k (\theta) = \sum_{i=1}^{s} h^i(V_{f\gamma}) \left[\sum_{n=1}^{m} \frac{\partial U_i}{\partial y_n} \frac{\partial f_n}{\partial \theta_k} + \frac{\partial U_i}{\partial \theta_k} \right] ,$$

$$k = 1,\ldots,s .$$

$L_{f\gamma}$ invertible implies that $\bar{L}_{f\gamma}$ is invertible. To invert $\bar{M}_{f\gamma}$, we must make $\|\bar{M}_{f\gamma} - \bar{L}_{f\gamma}\|$ sufficiently small. To do this, we see that

$$(11) \qquad \left\| \sum_{n=1}^{m} \frac{\partial U_i}{\partial y_n} \frac{\partial f_n}{\partial \theta_k} + \frac{\partial U_i}{\partial \theta_k} \right\|_o \leq \sum_{n=1}^{m} \left\| \frac{\partial U_i}{\partial y_n} \right\|_o r + \left\| \frac{\partial U_i}{\partial \theta_k} \right\|_o .$$

$$(12) \qquad \left\| \frac{\partial}{\partial \theta_t} \left[\sum_{n=1}^{m} \frac{\partial U_i}{\partial y_n} \frac{\partial f_n}{\partial \theta_k} + \frac{\partial U_i}{\partial \theta_k} \right] \right\|_o \leq \sum_{n=1}^{m} \left[\sum_{j=1}^{m} \left\| \frac{\partial^2 U_i}{\partial y_n \partial y_j} \right\|_o r^2 \right.$$

$$\left. + \left\| \frac{\partial^2 U_i}{\partial \theta_t \partial y_n} \right\|_o r + \left\| \frac{\partial U_i}{\partial y_n} \right\|_o u \right] + \left\| \frac{\partial^2 U_i}{\partial \theta_t \partial \theta_k} \right\|_o ,$$

$$(13) \qquad \| h(V_{f\gamma}) \|_o = \| h \|_o ,$$

and

$$\left\| \frac{\partial}{\partial \theta_k} h(V_{f\gamma}) \right\|_o \leq \|h\|_o \left[1 + \sum_{n=1}^{m} \left\| \frac{\partial U_i}{\partial y_n} \right\|_o r + \left\| \frac{\partial U_i}{\partial \theta_k} \right\|_o \right] .$$

$U(0, \theta, 0) = 0$, and $\frac{\partial U}{\partial y_n}$, $\frac{\partial^2 U}{\partial y_n \partial y_j}$ are uniformly bounded.

(iii) We can pick $r_2 \leq r_1$, $\gamma_3 \leq \gamma_2$, and u_1, so that if f is in $C(r_2, u_1)$ and $|\gamma| \leq \gamma_3$, then $\bar{M}_{f\gamma}$ is invertible and $\|\bar{M}_{f\gamma}^{-1}\| \leq B$.

We now have (from equation (8))

$$(14) \qquad \frac{\partial g}{\partial \theta_k} = \bar{M}_{f\gamma}^{-1} \left[\frac{\partial N}{\partial \theta_k} g + \frac{\partial W}{\partial y} \frac{\partial f}{\partial \theta_k} + \frac{\partial W}{\partial \theta_k} \right] .$$

Therefore

$$(15) \qquad \left\| \frac{\partial g}{\partial \theta_k} \right\| \leq B \left[\left\| \frac{\partial N}{\partial \theta_k} \right\| \|g\| + \left\| \frac{\partial W}{\partial y} \right\| \left\| \frac{\partial f}{\partial \theta_k} \right\| + \left\| \frac{\partial W}{\partial \theta_k} \right\| \right] .$$

(iv) We pick $r_3 \leq r_2$, $\gamma_4 \leq \gamma_3$, so that $B\| \frac{\partial W}{\partial y} \| \leq (\frac{1}{2})$, and so that

$$B\left[\left\|\frac{\partial N}{\partial \theta_k}\right\| r_3 + \left\|\frac{\partial W}{\partial \theta_k}\right\|\right] \leq \left(\frac{1}{2}\right) u_1 .$$

It follows that if f is in $C(r_3, u_1)$, and if $|\gamma| \leq \gamma_4$, then

$$(16) \qquad B\left[\left\|\frac{\partial N}{\partial \theta_k}\right\| r_3 + \left\|\frac{\partial W}{\partial \theta_k}\right\|\right] \leq \left[1 - B\left\|\frac{\partial W}{\partial y}\right\|\right] u_1 .$$

This implies that $\left\|\frac{\partial g}{\partial \theta_k}\right\| \leq u_1$.

We have selected a conditionally compact set that is mapped into itself by $L^{-1}W$. We will now show that $L^{-1}W$ also maps $\overline{C}$, the closure of $C(r_3, u_1)$, into itself and is continuous on $\overline{C}$.

If f_n is in $C(r_3, u_1)$, and if $\lim f_n = f_o$, then we can assume that $\lim g_n = g_o$, where g_n is the image of f_n. f_o and g_o are in $\overline{C}$. Let $L_n = L_{f_n\gamma}$, and $V_n = V_{f_n\gamma}$. Since $L_n g_n = W(f_n, \gamma)$, we have

$$(17) \qquad g_n(V_n(\theta)) = W(f_n(\theta), \theta, \gamma) + N(\theta) g_n(\theta).$$

The limit of the right hand side of the equation defines a function g^* .

$$(18) \qquad |g^*(\theta) - g_o(V_o(\theta))| \leq |g^*(\theta) - g_n(V_n(\theta))|$$

$$+ |g_n(V_o(\theta)) - g_o(V_o(0))| + |g_n(V_n(\theta)) - g_n(V_o(\theta))| .$$

Since g_n is in $C(r_3, u_1)$ (and therefore in $\overline{C}$),

$$(19) \qquad |g_n(V_n(\theta)) - g_n(V_o(\theta))| \leq s r_3 \| U(f_n, \gamma) - U(f_o, \gamma)\|_o .$$

By taking n large, we see that $\| g^* - g_o(V_o)\|_o = 0$. Therefore $g^* = g_o(V_o)$, and $g_o = L_o^{-1}W(f_o, \gamma)$. We have shown that $\overline{C}$ is mapped into itself. If we duplicate the argument, taking f_n in $\overline{C}$, we see that $L^{-1}W$ is continuous.

For fixed γ, $L^{-1}W$ is a continuous map of a compact convex set into itself. By Schauder's theorem, there is at least one

fixed point. We will complete the proof of our theorem by showing that this fixed point is unique, and that it is a continuous function of γ.

Suppose we have two invariant functions f and h. Then $L_{f\gamma}f = W(f,\gamma)$, and $L_{h\gamma}h = W(h, \gamma)$. Following Diliberto, we write

$$(20) \quad \| L_{f\gamma}f - L_{f\gamma}h\| \leq \| L_{h\gamma}h - L_{f\gamma}h\| + \| W(f, \gamma) - W(h, \gamma)\| .$$

Since

$$\| L_{f\gamma}^{-1}\| \leq A, \quad (\tfrac{1}{A}) \ \| f - h\| \leq \| L_{f\gamma}(f - h)\| .$$

$L_{h\gamma}h(\theta) - L_{f\gamma}h(\theta) = h(V_{h\gamma}(\theta)) - h(V_{f\gamma}(\theta))$. h is an element of $\overline{C}$, therefore h is Lipschitzian with constant r_3, and $\frac{\partial h}{\partial \theta_k}$ is Lipschitzian with constant u_1. Using this plus the big O properties of W and U,

(v) we pick $r_4 \leq r_3$, $u_2 \leq u_1$, and $\gamma_5 \leq \gamma_4$, so that the right hand side of (20) is less than or equal to $(\frac{1}{2A}) \ \| f - h\|$.

We now have that if $L^{-1}W$ is restricted to the closure of $C(r_4, u_2)$ and if $|\gamma| \leq \gamma_5$, the fixed points h and f satisfy the inequality

$$(21) \qquad \left(\frac{1}{A}\right) \ \| f - h\| \leq \left(\frac{1}{2A}\right) \ \|f - h\| .$$

Therefore f = h.

If we let $L_{f\gamma}f = W(f, \gamma)$, and $L_{h\gamma'}h = W(h, \gamma')$, then using (20), it is easy to show that $\|f - h\|$ tends to zero as $|\gamma - \gamma'|$ tends to zero. This completes the proof of the theorem.

Section 3

It was stated earlier that M. Marcus' result is included in the above theorem. Marcus assumed that $\|N\|_0 < 1$, and obtained an invariant function that is continuous and satisfies

a Lipschitz condition. In order to obtain invariant C^1 functions whose derivatives satisfy a Lipschitz condition, we will make the slightly stronger assumption that $\|N\| < 1$.

If this is done, $L_{f\gamma}^{-1}$ exists and is uniformly bounded. This is established by taking $\theta = G_{f\gamma}(\bar{\theta})$, the inverse of $\bar{\theta} = V_{f\gamma}(\bar{\theta})$, and then picking r_o and γ_o so that for all h, $\|N(G_{f\gamma})h(G_{f\gamma})\| \leq q\|h\|$. q is a constant such that $\|N\| < q < 1$.

Section 4

We now take s = 1. All functions considered will have period 1 in θ. Let B be a constant matrix, $Q(\theta)$ a non-singular periodic matrix, and let

$$(22) \quad N(\theta) = Q(\theta + \pi)BQ^{-1}(\theta), \quad \text{so} \quad N(\theta - \pi) = Q(\theta)BQ^{-1}(\theta - \pi).$$

We take $L_{f\gamma}$ as a transformation on the space of <u>complex</u> valued functions. If $L_{f\gamma}^{-1}$ exists and is uniformly bounded, it induces a transformation on the space of real functions that is also uniformly bounded. We must use the complex space since the eigenvalues of B play an important role.

Hufford showed that his $N(\theta)$ had the above form. He <u>assumed</u> that if λ is an eigenvalue of B, then $|\lambda| \neq 1$. We will use the equation $B = RJR^{-1}$, where R is a constant matrix and J is the Jordan canonical form of B.

If $N(\theta)$ has this special form, then $L_{f\gamma}^{-1}$ is uniformly bounded. To show this, we will find transformations acting independently on certain subspaces. Since none of the eigenvalues of B have absolute value equal to 1, these transformations will have inverses. This will imply that $L_{f\gamma}^{-1}$ exists and is uniformly bounded.

Let $P(\theta) = Q(\theta)R$. We define linear transformations $\bar{P}$, π^{-1}, $\bar{V}_{f\gamma}$ and $\bar{J}$ by

$$[\bar{P}h](\theta) = P(\theta)h(\theta)$$
$$[\pi^{-1}h](\theta) = h(\theta - \pi)$$
$$[\bar{V}_{f\gamma}h](\theta) = h(V_{f\gamma}(\theta))$$
$$[\bar{J}h](\theta) = Jh(\theta) \quad .$$

Then

(24) $\pi^{-1}L_{f\gamma} = \pi^{-1}\overline{V}_{f\gamma} - \overline{P}\overline{J}\pi^{-1}\overline{P}^{-1} = \overline{P}\,[\overline{P}^{-1}\,\pi^{-1}\overline{V}_{f\gamma}\overline{P} - \pi^{-1}\overline{J}]\overline{P}^{-1}$

Let

(25) $$K_{f\gamma} = \overline{P}^{-1}\,\pi^{-1}\overline{V}_{f\gamma}\overline{P} - \overline{J}\pi^{-1}\ .$$

(26) $[K_{f\gamma}h](\theta) = P^{-1}(\theta)P(V_{f\gamma}(\theta - \pi))h(V_{f\gamma}(\theta - \pi)) - Jh(\theta - \pi).$

(27) $$V_{f\gamma}(\theta - \pi) = \theta + U(f(\theta - \pi),\ \theta - \pi,\ \gamma)\ .$$

Because of this, if

(28) $$P(V_{f\gamma}(\theta)) - P(\theta) = P_{f\gamma}(\theta),$$

then $\|P_{f\gamma}\|$ can be made small by restricting $\|f\|$ and $|\gamma|$.

(29) $[K_{f\gamma}h](\theta) = h(V_{f\gamma}(\theta - \pi)) - Jh(\theta - \pi) + P^{-1}(\theta)P_{f\gamma}(\theta)h(V_{f\gamma}(\theta-\pi)).$

The first two terms define a linear transformation that acts independently on the subspaces determined by the canonical form of J. The third term can be made small in norm. A slight modification of the argument of Section 3 plus the proof used by Hufford in Lemma 3.1 of his thesis [1] shows that $\pi^{-1}\overline{V}_{f\gamma} - \pi^{-1}\overline{J}$ has a uniformly bounded inverse. This implies that $L_{f\gamma}^{-1}$ is uniformly bounded.

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